Supplemental Notes

EE503 Week 01 Dr. Franzke

Probability for Electrical and Computer Engineers.

HW # 1

1. HW # 1 handout
2. Leon Garcia
3. Gubner

Topics

- Truth table proofs
- IMAC (format for exam)
- Probability Spaces CUT CAT
- Set functions $f(A) = \{ f(x) \in Y : x \in A \}$.

"pull-back" $f^{-1}(B) = \{ x \in X : f(x) \in B \}$

Theorem (1) $P \,\&\, Q \implies P$

(2) $P \implies P \vee Q$

<u>Prf</u>: by (binary) truth table.

P	Q	P	$\&$	Q	$\implies$	P	P	$\implies$	P	$\vee$	Q
1	1	1	1	1	1	1	1	1	1	1	1
0	1	0	0	1	1	0	0	1	0	1	1
1	0	1	0	0	1	1	1	1	1	1	0
0	0	0	0	0	1	0	0	1	0	0	0

QED.

IMAC format

I: Issue

- <u>associative</u> memory ("spot the issue")

M: Math rule

- rote <u>memorization</u>

A: Apply math to facts (Analysis)

- pattern matching
- element because fact.

C: Conclusion

- deductive logic

IMAC example: Finite sigma-algebra

Sample space X contains four elements $X = \{a, b, c, d\}$. So its power set or set of subsets 2^X contains 2^4 elements. Thus the power set 2^X gives rise to 2^{16} subcollections of sets. Suppose that $A \subset 2^X$ is one of these 2^{16} set subcollections of 2^X and that A contains the following 7 sets:

$$A = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

Can we define a probability space (X, A, P) using the set collection A if P is a probability measure?

(use IMAC format to answer)

Soln:

$\underline{\text{I}}$: sigma-algebra (s.a.)

$\underline{\text{M}}$: $\mathcal{A} \subset 2^X$ is a sigma-algebra iff $\mathcal{A}$ is $\underline{\text{CUT}}$:

$$\begin{cases} C: & A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{A} \\ U: & A_1 \in \mathcal{A}, A_2 \in \mathcal{A}, \ldots \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A} \\ T: & X \in \mathcal{A} \end{cases}$$

$\underline{\text{A}}$:

$\underline{\text{C}}$:

$\{a\}^c = \{b,c,d\} \in \mathcal{A}$

$\{b\}^c = \{a,c,d\} \in \mathcal{A}$.

$\{a,b\}^c = \{c,d\} \notin \mathcal{A}$

$$\left(\begin{array}{l} \{b,c,d\}^c = \{a\} \in \mathcal{A} \\ \{a,c,d\}^c = \{b\} \in \mathcal{A} \\ \phi^c = X \in \mathcal{A}, \quad X^c = \phi \in \mathcal{A} \end{array} \right)$$

$\therefore$ $\mathcal{Q}$ is not closed under complement

because $\{a,b\}^c \notin \mathcal{Q}$

U: $\{a\} \cup \{b\} = \{a,b\} \in \mathcal{Q}$

$\{a\} \cup \{a,b\} = \{a,b\} \in \mathcal{Q}$

$\{a\} \cup \{a,c,d\} = \{a,c,d\} \in \mathcal{Q}$

$\{a\} \cup \{b,c,d\} = X \in \mathcal{Q}$

$\{b\} \cup \{a,b\} = \{a,b\} \in \mathcal{Q}$

$\{b\} \cup \{a,c,d\} = X \in \mathcal{Q}$

$\{b\} \cup \{b,c,d\} = \{b,c,d\} \in \mathcal{Q}$

$\emptyset \cup S = S \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$X \cup S = X \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$\therefore$ $\mathcal{Q}$ is closed under union.

T: $X \in \mathcal{Q}$.

$\therefore$ $\mathcal{Q}$ is $\cancel{C}$UT

$\therefore$ $\mathcal{Q}$ is not a sigma-algebra.

C: **NO.**

(X, A, P) cannot be a probability space

because A is not a sigma-algebra.

CUT CAT

- acronym (mnemonic) for a probability space

① CUT

- definition of a sigma-algebra

Space X ("Sample space" Ω)

$2^X = \{A : A \subset X\}$ - power set
- set of all subsets.

$\therefore$ $x \in A \subset X \in \mathcal{Q} \subset 2^X$.

if $\mathcal{Q}$ is a subcollection of 2^X (that contains X)

set of sets

Defn: $\mathcal{Q} \subset 2^X$ is a sigma-algebra (s.A. or σ-algebra)
iff $\mathcal{Q}$ is CUT:

C: closed under complement.

$$A \in \mathcal{Q} \implies A^c \in \mathcal{Q}$$

U: closed under countable unions

$$A_1 \in \mathcal{Q},\ A_2 \in \mathcal{Q}, \ldots \implies \bigcup_k A_k \in \mathcal{Q}$$

T: total set X

$$X \in \mathcal{Q}$$

Defn: A is an event (is "measurable") iff
$A \in \mathcal{Q}$ and $\mathcal{Q}$ is a S.A.

Defn: $(X, \mathcal{Q})$ is a measurable space iff
$\mathcal{Q} \subset 2^X$ is a S.A.

② CAT

- definition of a probability measure

Defn: Suppose $(X, \mathcal{Q})$ is a measurable space.
Then $P : \mathcal{Q} \longrightarrow [0,1]$ is a probability measure iff P is CAT.

CA: Countable Additive

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k) \quad \text{if} \quad A_i \cap A_j = \phi \text{ when } i \neq j$$

(A_i and A_j are "mutually exclusive")

T: Total set X gets unit mass:

$$P(X) = 1$$

Addition Theorem

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Set functions

points to points : $f : X \longrightarrow Y$

$x \longmapsto f(x)$

sets to sets $f : 2^X \longrightarrow 2^Y$

$f^{-1} : 2^Y \longrightarrow 2^X$

Defn: Set image $f(A) = \{ f(x) : x \in A \}$.

Defn: Inverse image ("pullback")

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}.$$

Note: set inverse $f^{-1} : 2^Y \longrightarrow 2^X$ always exists.

Defn: $(X, \mathcal{Q}, P)$ is a probability space iff

$\mathcal{Q} \subset 2^X$ is CUT and

$P : \mathcal{Q} \longrightarrow [0,1]$ is CAT

$\therefore$ CUT CAT

Thrm. $f(A \cup B) \subset f(A) \cup f(B)$

Prf: pick $y = f(x) \in f(A \cup B)$ Assume

$(\exists x \in X : y = f(x)$ and $x \in A \cup B)$

$\therefore x \in A \cup B$ def f

$\therefore x \in A$ OR $x \in B$ def $\cup$

$\therefore f(x) \in f(A)$ OR $f(x) \in B.$ def f

$\therefore f(x) \in f(A) \cup f(B).$ def $\cup$

$\therefore f(A \cup B) \subset f(A) \cup f(B)$ def $\subset$

QED.

Thrm: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Prf:

Claim 1: $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Prf:

pick $x \in f^{-1}(A \cup B)$ Assume

$\therefore f(x) \in A \cup B$ def f^{-1}

$\therefore f(x) \in A$ OR $f(x) \in B.$ def $\cup$

$\therefore x \in f^{-1}(A)$ OR $x \in f^{-1}(B)$ def f^{-1}

$\therefore x \in f^{-1}(A) \cup f^{-1}(B)$ def $\cup$

$\therefore f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$ def $\subset$

QED (claim 1)

Claim 2: $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Prf:

pick $x \in f^{-1}(A) \cup f^{-1}(B)$	Assume
$\therefore$ $x \in f^{-1}(A)$ OR $x \in f^{-1}(B)$	def $\cup$
$\therefore$ $f(x) \in A$ OR $f(x) \in B$	def f^{-1}
$\therefore$ $f(x) \in A \cup B$	def $\cup$
$\therefore$ $x \in f^{-1}(A \cup B)$	def f^{-1}
$\therefore$ $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$	def $\subset$

QED (Claim 2)

$\therefore$ Claim 1 & Claim 2 $\Longrightarrow$

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ def $=$

QED

Thrm: (Distributivity)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Prf:

Lemma: $P \,\&\, (Q \vee R) = (P \,\&\, Q) \vee (P \,\&\, R)$

Prf: by truth table, propositional calculus

P	Q	R	P	&	(Q	∨	R)	=	(P	&	Q)	∨	(P	&	R)
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	0	0	1	0	0	0	1
1	0	1	1	1	0	1	1	1	1	0	0	1	1	1	1
0	0	1	0	0	0	1	1	1	0	0	0	0	0	0	1
1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	0
0	1	0	0	0	1	1	0	1	0	0	1	0	0	0	0
1	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

∴ Valid ("tautology")

∴ QED (Lemma)

Claim 1: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

Prf: pick $x \in A \cap (B \cup C)$ — Assume

∴ $x \in A$ AND $x \in (B \cup C)$ — def $\cap$

∴ $\underbrace{x \in A}_{P}$ AND $(\underbrace{x \in B}_{Q}$ OR $\underbrace{x \in C}_{R})$ for fixed x — def $\cup$

∴ $(x \in A \,\&\, x \in B) \vee (x \in A \,\&\, x \in C)$ — lemma

$\therefore (x \in A \cap B) \vee (x \in A \cap C)$ defn $\cap$

$\therefore x \in (A \cap B) \cup (A \cap C)$ defn $\cup$

$\therefore A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ defn $\subset$

QED (claim 1)

Claim 2: $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

Pf:

pick $x \in (A \cap B) \cup (A \cap C)$ Assume

$\therefore (x \in A \cap B) \vee (x \in A \cap C)$ def $\cup$

$\therefore (x \in A \ \& \ x \in B) \vee (x \in A \ \& \ x \in C)$ def $\cap$

$\therefore x \in A \ \& \ (x \in B \vee x \in C)$ lemma

$\therefore x \in A \ \& \ x \in (B \cap C)$ def $\cup$

$\therefore x \in A \cap (B \cup C)$ def $\cap$

$\therefore (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ def $\subset$

QED (claim 2)

$\therefore$ Claim 1 and Claim 2 $\Longrightarrow$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ def $=$

QED.

Thrm: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(similar lemma and proof)